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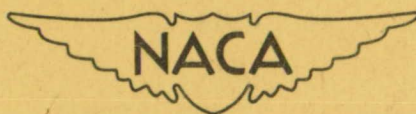
TECHNICAL NOTE 1943

THEORETICAL RELATIONS BETWEEN THE STABILITY DERIVATIVES
OF A WING IN DIRECT AND IN REVERSE
SUPERSONIC FLOW

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S U M M A R Y

An investigation is made by means of the linearized theory of the relations between the supersonic stability derivatives of a wing and its reverse obtained by interchanging the leading and trailing edges. The analysis includes wings that have a supersonic leading edge, a supersonic trailing edge, and either supersonic side edges or nonre-entrant subsonic side edges which are noninteracting, that is, neither edge lies in the zone of action of the other. The investigation is made for thin flat-plate wings and includes steady vertical and longitudinal motions, steady rolling and pitching, and motions in which the wing is accelerating uniformly in the vertical direction.

The results of the investigation are obtained with respect to body axes and indicate that the rates of change of lift and drag with angle of attack and forward speed, the damping in roll, and the rate of change of lift with a uniform variation of angle of attack with time were all equal for the wing and its reverse. Also, with respect to axes of rotation that are fixed with respect to the wing configuration, the damping in pitch was equal for the wing and its reverse; whereas, the rate of change of pitching moment with angle of attack for the wing was equal to the rate of change of lift with pitching velocity multiplied by the stream velocity for the reverse of the wing.

I N T R O D U C T I O N

The theorem of flow reversibility as demonstrated by Von Kármán (reference 1) states that, within the validity of the linearized theory, the wave drag of an uncambered wing at zero lift is the same as that of its reverse obtained by interchanging the leading and trailing edges. The theorem of flow reversibility was later extended by Hayes (reference 2) to apply to the rate of change of lift with angle of attack for wing configurations in which all the edges are supersonic. (The term

"supersonic edge" as used herein refers to an edge for which the normal component of the stream velocity is supersonic; a "subsonic edge" is defined in an analogous manner.) For this type of wing configuration, Flax (reference 3) obtained relations between a wing and its reverse for the rate of change of lift and pitching moment with angle of attack, for the rate of change of lift and pitching moment with pitching, and for the damping in roll. A treatment of the flow-reversibility theorem for the rate of change of lift with angle of attack has been reported in reference 4 for wings which have a supersonic leading edge and supersonic trailing edge and nonre-entrant subsonic edges which are noninteracting. A nonre-entrant edge is one for which no part of the trailing vortex sheet impinges on another part of the airfoil. Non-interacting subsonic edges are such that neither edge lies in the zone of action of the other.

In the present paper, simple relations between the stability derivatives of a wing and its reverse in supersonic flow are derived by means of the linearized theory. The analysis includes wings that have a supersonic leading edge, supersonic trailing edge, and either supersonic side edges or nonre-entrant subsonic side edges that are noninteracting.

The analysis is made for thin flat-plate wings for steady vertical and longitudinal motions, for steady rolling and pitching, and for motions in which the wing is accelerating uniformly in the vertical direction. The analysis for the wings with side edges that act as a subsonic trailing edge (subsonic portions of raked-in wing tips) is based on the application of the Kutta-Joukowski condition.

S Y M B O L S

x, y, z	rectangular coordinates
ξ, η	auxiliary variables used to replace x and y , respectively
u, v	Mach coordinates used in appendix B (see fig. 6)
u_1, v_1	source point in Mach coordinate system
u, w	incremental velocities of wing along x - and z -axes, respectively; u is positive in flight direction; w is positive downward

u_0, w_0 disturbance velocity of fluid along x- and z-axes, respectively; u_0 is positive in free-stream direction; w_0 is positive upward

$\Delta u_0'$ contribution to u_0 for steady motions in reverse flow of vorticity shed in wake of portion of subsonic tip which is raked inward (based on application of Kutta-Joukowski condition)

$\delta u_0', \delta L'$ partial contribution of sources in an elemental area to velocity u_0' at a point and to elementary lift over an area, respectively

V free-stream velocity

a speed of sound; also, used to refer to a line and, with prime, to refer to a line for wing in reverse flow (see figs. 2 to 4)

M stream Mach number (V/a)

$$B = \sqrt{M^2 - 1}$$

$$R = \sqrt{(x - \xi)^2 - B^2(y - \eta)^2}$$

μ Mach angle

ϕ disturbance-velocity potential on upper surface of airfoil

$$\phi_x = \frac{\partial \phi}{\partial x}$$

t time following disturbance

τ_1, τ_2 time-lag parameters defined by equations (31) and (32), respectively

$$\phi_t = \frac{\partial \phi}{\partial t}$$

ρ mass density

Δp pressure difference between lower and upper surfaces of airfoil, positive in direction of lift

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- $\Delta(\Delta P), \Delta(\Delta P)'$ contribution to ΔP in uniform accelerated vertical motion in direct flow and in reverse flow, respectively, of vorticity shed in wake of portion of subsonic tip which is raked inward (based on application of Kutta-Joukowski condition)
- C, D, E, F, G regions of integration for obtaining contribution to ϕ , ϕ_x , or u_0 (see fig. 2); also used as subscripts for limits of integration in appendix A
- A combined regions C, D, E
- P, Q points in a region (see fig. 2)
- $L(\xi_{LE})$ function representing equation for leading edge (η_{LE})
- $\bar{c}$ mean chord
- S wing area
- b wing span
- γ local inclination of wing tip to x-axis at point of intersection of line a or a' (fig. 3 or 4) with wing tip; positive when tip slopes to right
- i point of intersection of line a with side edge in figure 4
- α wing angle of attack in steady flight (w/V)
- $\dot{\alpha}$ derivative of α with respect to time
- p, q angular velocities about x- and y-axes, respectively
- D, L drag and lift forces, respectively
- D_{wo} wave drag at zero lift
- M, L_r pitching and rolling moments, respectively
- $C_D, C_{D_{wo}}, C_L$ drag, wave drag at zero lift, and lift coefficients, respectively $\left(\frac{D, D_{wo}, L}{\frac{\rho}{2} V^2 S} \right)$
- C_m pitching-moment coefficient $\left(\frac{M}{\frac{\rho}{2} V^2 S \bar{c}} \right)$

C_l	rolling-moment coefficient $\left(\frac{L_r}{\frac{\rho}{2} V^2 S b} \right)$
ψ	additional potential associated with $\Delta(\Delta P)$ or each of integrals in equation (C3)
τ	additional potential associated with integral in equation (C4)
K	constant used in equation (C3)
F	constant used in equation (C4)
E	constant used in equation (C6)
x'	auxiliary variable of integration which replaces x
Subscripts:	
1,2,3	contribution of component terms of equation (3) to $\delta u_0'$ and $\delta L'$
LE	refers to wing leading edge

Primes used with disturbance velocities, forces, moments, angular rotations, and stability derivatives refer to wing in reverse flow.

Whenever α , q , u , p , and $\dot{\alpha}$ are used as subscripts, a nondimensional derivative is indicated and this derivative is the slope through

zero. For example, $C_{m_q} = \left[\frac{\partial C_m}{\partial \left(\frac{q \bar{c}}{2V} \right)} \right]_{q \rightarrow 0}$, $C_{L_u} = \left[\frac{\partial C_L}{\partial \left(\frac{u}{\bar{V}} \right)} \right]_{u \rightarrow 0}$,

$C_{l_p} = \left[\frac{\partial C_l}{\partial \left(\frac{p b}{2V} \right)} \right]_{p \rightarrow 0}$, and $C_{L_{\dot{\alpha}}} = \left[\frac{\partial C_L}{\partial \left(\frac{\dot{\alpha} \bar{c}}{2V} \right)} \right]_{\dot{\alpha} \rightarrow 0, t \rightarrow 0}$.

ANALYSIS

The analysis is based on the usual assumptions for the linearized theory of supersonic flow, that is, isentropic and nonviscous flow, small disturbance velocities relative to the free-stream velocity, and a constant velocity of sound. The wing is assumed to be thin and flat and to lie in the xy -plane ($z = 0$).

STEADY MOTIONS

Derivation of Basic Equations

If the foregoing assumptions are made, the velocity potential ϕ at any point (x,y) in the plane of the wing for steady motion may be expressed in the following form (see, for example, reference 5):

$$\phi(x,y) = -\frac{1}{\pi} \iint_R \frac{w_0(\xi,\eta) d\xi d\eta}{R} \quad (1)$$

where $R = \sqrt{(x - \xi)^2 - B^2(y - \eta)^2}$ and where the integration extends over the entire region in the plane of the wing that can influence the point (x,y) . The quantity $w_0(\xi,\eta)$ is the fluid perturbation velocity in the vertical direction at the point $(\xi,\eta,0)$. Equation (1) expresses the condition that a solution for ϕ that satisfies the basic linearized partial-differential equation of the flow and the required boundary conditions may be obtained by superposing elementary source solutions of strength proportional to w_0 over the region of integration.

The present analysis is limited to plan forms that have a supersonic leading edge, a supersonic trailing edge, and either supersonic side edges or nonre-entrant subsonic side edges that are noninteracting. A general plan form that fulfills these conditions is shown in figure 1. The typical region that can influence a point (x,y) is contained in the xy -plane within the Mach forecone from the point. Figure 2 illustrates the typical Mach forecones or regions of influence for two points P and Q for a wing with a supersonic leading edge, a supersonic trailing edge, and a streamwise tip. The Mach forecone may intersect only the leading edge, as for point P in figure 2, or it may intersect the side edge as well, as for point Q in the same figure. The treatment of the second situation is the more complicated and includes the first situation as a special case. The second situation is shown in detail in figure 2. Thus the component regions that influence the point Q are C , D , E , F , and G . It has been

shown in reference 5 that, for a thin flat-plate wing in certain cases, the influence of the external field F on the point (x,y) is identically canceled by the effect of region G in the wing. This cancellation is valid under the following conditions:

1. The vorticity in the wake of the wing cannot influence any part of the wing.
2. The regions external to the wing that are influenced by subsonic edges must not interact with each other in such a way as to influence any part of the wing (noninteracting subsonic edges).

Restriction 1 limits the validity of the cancellation to portions of a subsonic side edge which are either in the streamwise direction or raked out. Those portions of a subsonic side edge which are raked in, however, will produce a wake in which the vorticity will influence portions of the wing. (See fig. 1 or 3(b).)

If the point (x,y) (fig. 2) lies inboard and out of the tip region, the exterior field F disappears and the regions of influence for the point are bounded by the wing boundary and the Mach lines from its forecone. The component regions that influence the point thus degenerate into two regions which correspond to C and E (fig. 2). These component regions C and E also correspond to those that influence any point in a wing having supersonic side edges that are raked in or out. Thus, the supersonic raked tip may be treated simply as a continuation of the leading-edge wing boundary.

Because of this degeneration of the component regions C , D , and E , the subsequent analysis refers specifically to points on the wing that are influenced by subsonic tips. The results obtained for this case can then be made to apply also to the region between the tip Mach cones for wings with subsonic tips and to the entire wing for the case of supersonic tips of arbitrary rake by simply inserting the condition that the component region D vanishes for these cases.

For simplicity, the analysis that follows immediately refers to figure 2 in which the tip is shown to be straight and in the direction of the stream. The general analysis, however, applies to all planforms in which the cancellation of the region external to the wing is valid.

The pressure at the point (x,y) is proportional to the quantity $\phi_x(x,y)$ that may be obtained by differentiating equation (1) with respect to x . The differentiation of equation (1) with respect to x under the integral sign must be performed with special attention to the variable limits for ξ and η . (See fig. 2.) If this

differentiation is performed, the contribution of each component region for $\phi(x,y)$ (regions C, D, and E in fig. 2) results in five integrals. One of the integrals for each region is a surface integral; whereas the remaining four are integrals that are to be evaluated along the boundary of the region. These five integrals for $\phi_x(x,y)$ are given in appendix A.

As a result of an analysis that is given in appendix A, the foregoing integrals for the component regions C, D, and E, which are hereinafter denoted by A, may be written in the following form:

$$u_0(x,y) = \phi_x(x,y)$$

$$= \frac{1}{\pi} \left[\iint_A \frac{w_0(\xi, \eta)(x - \xi) d\xi d\eta}{R^3} - \frac{1}{\pi} \left(\frac{\partial \eta}{\partial x} 2\pi \right) \int_a \frac{w_0(\xi, \eta) d\xi}{R} - \frac{w_0(x,y)}{B} \right] \quad (2)$$

The notation $\left[\right]$ designates the "finite part of the infinite integral," a concept introduced by Hadamard (reference 6). For the cases considered in this analysis in which w_0 is constant, is any function of ξ , or varies linearly with η , if the integration is performed first with respect to η , it has been verified that the η limits for the surface integral along boundaries of the Mach forecone from the point must be omitted. More generally, however, the surface integral in equation (2) for an arbitrary variation of w_0 with η can be evaluated as the finite part of the integral by the method introduced by Hadamard (reference 6) and elaborated in reference 7.

Elementary Lifts for Wing in Direct Flow and in Reverse Flow

Equation (2), when modified for the wake effect contributed by portions of a subsonic side edge that are raked inward is the basic equation for deriving relationships between the stability derivatives of a wing and its reverse for steady motion. In the derivation, the concept of a forezone of influence and an afterzone of influence of a point is used extensively. Thus, consider the wing shown in figure 3(a) in direct flow. As noted previously, only the sources in region A can influence the velocity u_0 at P. Thus A is called the forezone of point P. Now consider the forezones of type A for various points P' below P, which are located so that the forezones from these points include the point P. The point P' is then found to be limited to the area A' in figure 3(a). Thus the area A' defines the region of

disturbance produced by a source element at P. The region A' therefore is called the afterzone of influence of point P, or the afterzone of point P.

A simple relationship is subsequently shown to exist between the integrated effect of the sources in region A on the elementary area at P in direct flow and the integrated effect of the sources in the elementary area at P on the region A in reverse flow.

In figure 3(a) for direct flow, let the source points be tentatively designated as (ξ, η) and field points as (x, y) . If P is considered as a field point, then the velocity u_0 at P or (x, y) is given by equation (2). The following relationship can be obtained from figures 2 and 3(a):

$$\frac{\partial \eta_{2D}}{\partial x} = - \frac{1}{B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right)$$

where γ represents the local inclination of the wing tip to the x-axis at the point of intersection of line a with the tip. Equation (2) for direct flow then becomes

$$\begin{aligned} u_0(x, y) &= \phi_x(x, y) \\ &= \frac{1}{\pi} \left[\iint_A \frac{w_0(\xi, \eta)(x - \xi) d\xi d\eta}{R^3} \right. \\ &\quad \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{w_0(\xi, \eta) d\xi}{R} - \frac{w_0(x, y)}{B} \right] \end{aligned} \quad (3)$$

Suppose now that the direction of the stream flow is reversed. For this reverse flow, it is convenient for the purpose of later developments to reverse the notation for source points and field points. Thus,

Direct flow	Reverse flow
Source point: (ξ, η)	Source point: (x, y)
Field point: (x, y)	Field point: (ξ, η)

Consider the wing in figure 3(a) when the direction of flow is reversed and let P be a field point. Then, relative to the reverse-flow direction, the line a' intersects the side edge at a point at

which the local slope of the side edge is inward (raked-in tip portion). This portion of the wing tip acts as a subsonic trailing edge and the vorticity shed in its wake influences the wing. (See fig. 3(b).) The contribution of this vorticity or wake effect to the velocity u_o' for a point on the wing in reverse flow is designated as $\Delta u_o'$. This contribution $\Delta u_o'$ is determined on the basis that the Kutta-Joukowski condition is satisfied along the subsonic side edge. The contribution of the part other than the wake effect to the velocity u_o' at point P or (ξ, η) is obtained from equation (3) in three steps: First, the signs of ξ and x (also γ) are reversed (but not those of the element dx) because of the reversal of the direction of the x -axis relative to the flow direction. Next, the coordinates ξ, η and x, y are interchanged in each system because of the changed notation for source and field points in reverse flow. Finally, the area for the double integration is changed to A' and the path of the line integral is specified to be along a' . Thus, the velocity u_o' at point P or (ξ, η) in reverse flow is

$$u_o'(\xi, \eta) = \frac{1}{\pi} \iint_{A'} \frac{w_o'(x, y)(x - \xi) dx dy}{R^3} + \frac{1}{\pi B} \left(\frac{1 + B \tan \gamma}{1 - B \tan \gamma} \right) \int_{a'} \frac{w_o'(x, y) dx}{R} - \frac{w_o'(\xi, \eta)}{B} + \Delta u_o' \quad (4)$$

By means of an extension for generalizing the results obtained in reference 8, it may be shown that in the case of flat plates in which the side edges are raked in and are subsonic, if the Kutta-Joukowski condition is applied, the vorticity in the wake is of just sufficient strength to cancel the contribution to u_o of a line integral that yields an infinite pressure along the subsonic tip. This line integral, which is canceled by the wake effect, is evaluated on the wing along the line a' . Thus, it is found that, with regard to the reverse flow shown in figure 3(a), the effective forezone of influence for the point P or (ξ, η) is limited to the area A' when the Kutta-Joukowski condition is applied.

Again, by an extension of the results in reference 8, the contribution to u_o' of the vorticity in the wake or $\Delta u_o'$ is

$$\Delta u_o' = - \frac{2 \tan \gamma}{\pi(1 - B \tan \gamma)} \int_{a'} \frac{w_o'(x, y) dx}{R} \quad (5)$$

Substituting the expression for $\Delta u_0'$ from equation (5) into equation (4) and simplifying yields

$$u_0'(\xi, \eta) = \frac{1}{\pi} \iint_{A'} \frac{w_0'(x, y)(x - \xi) dx dy}{R^3} + \frac{1}{\pi B} \int_a \frac{w_0'(x, y) dx}{R} - \frac{w_0'(\xi, \eta)}{B} \quad (6)$$

In figure 4(a) is shown a portion of a wing in direct flow and in figure 4(b), in reverse flow. The wing is intended to meet the general conditions for the plan form illustrated in figure 1. In figure 4(a) for direct flow, the shaded region A is the forezone of the point (x, y) in the element dx dy. In figure 4(b), for reverse flow, the shaded region A is now the afterzone of the same point. Thus, in direct flow, the sources in region A induce lift on the element dx dy; whereas, in reverse flow, the sources in the element dx dy induce lift over region A. For the cases treated in this paper, a simple relationship is subsequently shown to exist between these two elementary lifts.

Elementary lift in direct flow.- For convenience in the analysis, let the point (x, y) in figure 4(a) in direct flow be such that the line a intersects the side edge at a point i at which the local slope of the side edge is outward (raked-out tip portion). This choice does not affect the generality of the subsequent analysis. The lift on the element dx dy in figure 4(a) can be evaluated at once. The velocity u_0 at dx dy is given by equation (3) and the lift δL according to Bernoulli's law is equal to $2\rho V u_0 dx dy$. Thus,

$$\delta L(dx dy) = 2\rho V dx dy \left[\frac{1}{\pi} \iint_A \frac{w_0(\xi, \eta)(x - \xi) d\xi d\eta}{R^3} + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{w_0(\xi, \eta) d\xi}{R} - \frac{w_0(x, y)}{B} \right] \quad (7)$$

Elementary lift in reverse flow.- In the reverse flow shown in figure 4(b), the point i is now located relative to this reverse-flow direction at a point at which the local slope of the side edge is inward (raked-in tip portion). For this case, therefore, equation (6) is used to determine the lift on region A due to the sources in dx dy. This evaluation is made by subdividing equation (6) into its elementary components, the contributions of which are then resummed in a different fashion to give the total lift on the region.

Consider first the contribution of the surface integral in equation (6) over region A' in figure 3(a) for reverse flow, where region A' is the forezone of the point P or (ξ, η) . The partial contribution of the sources in $dx dy$ (any element in region A') to the velocity u_0' at the point (ξ, η) is obtained from equation (6) by dropping the integral signs. Thus,

$$\delta_1 u_0'(\xi, \eta) = \frac{1}{\pi} \left[\frac{w_0'(x, y)(x - \xi) dx dy}{R^3} \right] \quad (8)$$

In figure 4(b) for reverse flow, therefore, each element $d\xi d\eta$ in region A experiences an incremental lift induced by the sources in the element $dx dy$ of amount $2\rho V d\xi d\eta$ times the quantity given in equation (8). The corresponding incremental lift for the whole of region A is then given by the integral

$$\delta_1 L' = \frac{2\rho V}{\pi} dx dy \left[\iint_A \frac{w_0'(x, y)(x - \xi) d\xi d\eta}{R^3} \right] \quad (9)$$

The lift contribution that results from the line integral along a' , which appears in equation (6), requires special consideration. The details are given in appendix B. The result is

$$\delta_2 L' = \frac{2\rho V}{B\pi} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) dx dy \int_a \frac{w_0'(x, y) d\xi}{R} \quad (10)$$

This incremental lift is to be regarded as localized in a strip of infinitesimal width along the line a . The third part of the contribution of the sources in $dx dy$ to the incremental lift on region A (fig. 4(b)) is obtained from the last term in equation (6). This term $-\frac{w_0'(\xi, \eta)}{B}$ expresses the contribution at a point only if the field point is in the immediate vicinity of the source point, that is, for $(x, y) \rightarrow (\xi, \eta)$. Thus, the point (x, y) in region A experiences an incremental contribution $\delta_3 u_0'$ because of the sources in $dx dy$ of amount

$$\delta_3 u_0'(x, y) = - \frac{w_0'(x, y)}{B}$$

The incremental lift due to this contribution is

$$\delta_3 L' = - \frac{2\rho V w_0'(x,y) dx dy}{B} \quad (11)$$

The total lift induced on the region A by the sources in the element $dx dy$ in reverse flow is given by the sum of $\delta_1 L'$, $\delta_2 L'$, and $\delta_3 L'$, which are evaluated in equations (9), (10), and (11), respectively. The total lift is

$$\begin{aligned} \delta L' = 2\rho V dx dy & \left[\frac{1}{\pi} \iint_A \frac{w_0'(x,y)(x - \xi) d\xi d\eta}{R^3} \right. \\ & \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{w_0'(x,y) d\xi}{R} - \frac{w_0'(x,y)}{B} \right] \quad (12) \end{aligned}$$

Reversibility Relations for Stability Derivatives

In the subsequent analysis all the stability derivatives are expressed with respect to body axes that are fixed with respect to the wing configuration.

In the analysis for the stability derivatives that involve moments and angular velocities the following remarks concerning notation are pertinent. In conventional notation, the positive direction for moments and angular velocities is defined with respect to the direction of flight. The flight direction is, in effect, reversed relative to the coordinate axes (fig. 4) for the wing in reverse flow. Thus, the sign for all moments and angular velocities is also reversed relative to these coordinate axes that are fixed in the wing. The positive direction for the several moments and angular velocities is indicated in figure 5. The analysis has dealt for convenience with the relative behavior of a given wing in direct and in reverse flow. However, a given wing in reverse flow and the reverse of a given wing in direct flow are exactly equivalent and their stability derivatives are identical.

Derivative C_{L_α} .-- Reconsider the significance of δL and $\delta L'$.

With reference to figure 4(a), δL expresses the lift induced on the element $dx dy$ by all the sources in the forezone A in direct flow. With reference to figure 4(b), $\delta L'$ expresses the total lift induced on the afterzone A by the sources in $dx dy$ in reverse flow. The area A is geometrically the same in both direct and reverse flow. The

expressions for δL and $\delta L'$ are now recapitulated for comparison:

$$\delta L = 2\rho V \, dx \, dy \left[\frac{1}{\pi} \left| \iint_A \frac{w_o(\xi, \eta)(x - \xi) d\xi \, d\eta}{R^3} \right. \right. \\ \left. \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{w_o(\xi, \eta) d\xi}{R} - \frac{w_o(x, y)}{B} \right] \quad (13)$$

and

$$\delta L' = 2\rho V \, dx \, dy \left[\frac{1}{\pi} \left| \iint_A \frac{w_o'(x, y)(x - \xi) d\xi \, d\eta}{R^3} \right. \right. \\ \left. \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{w_o'(x, y) d\xi}{R} - \frac{w_o'(x, y)}{B} \right] \quad (14)$$

If the wing is flat, then for the same constant angle of attack over the wing for both direct and reverse flow

$$w_o(\xi, \eta) = w_o'(x, y) = -V\alpha$$

over the entire wing. Thus, for this case, the elementary lifts δL and $\delta L'$ in equations (13) and (14), respectively, become identical. In both direct flow and reverse flow, the total wing lift is given by the integration of the elementary lift over the plan form, the element of integration being $dx \, dy$. Thus, the derivative $C_{L\alpha}$ for the wing in direct flow and reverse flow is

$$C_{L\alpha} = \frac{\iint_S \delta L \, dx \, dy}{\alpha \frac{\rho}{2} V^2 S} = C_{L\alpha}'$$

The theorem of reversibility is thus proved for the lift of wings of the general type illustrated in figure 1 or 4.

Derivatives $C_{m\alpha}$ and C_{Lq} .— If the wing in direct flow (fig. 4(a)) is at a constant angle of attack α , then

$$w_o(\xi, \eta) = -V\alpha \quad (15)$$

The pitching moment δM about the y -axis induced by the sources in region A on the element $dx dy$ is obtained by means of equation (13) and the value for w_0 given by equation (15). Thus,

$$\begin{aligned}\delta M(dx dy) &= -x\delta L \\ &= 2\rho V^2 \alpha x dx dy \left[\frac{1}{\pi} \left[\iint_A \frac{(x - \xi)d\xi d\eta}{R^3} \right. \right. \\ &\quad \left. \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{d\xi}{R} - \frac{1}{B} \right] \right] \quad (16)\end{aligned}$$

Now if the wing in reverse flow (fig. 4(b)) is pitching about the y -axis with an angular velocity of q , then

$$w_0'(x, y) = qx \quad (17)$$

(q is taken positive with respect to reverse-flow direction). The incremental lift $\delta L'$ on region A due to the sources in the element $dx dy$ in reverse flow contributed by this pitching angular velocity is obtained from equation (14) by substituting for w_0' the relation given by equation (17). Thus,

$$\delta L' = 2\rho Vqx dx dy \left[\frac{1}{\pi} \left[\iint_A \frac{(x - \xi)d\xi d\eta}{R^3} \right. \right. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{d\xi}{R} - \frac{1}{B} \left. \right] \quad (18)$$

A comparison of equations (16) and (18) shows that the elementary pitching moment $\delta M'$ differs from the elementary lift $\delta L'$ by a constant factor of proportionality, or

$$\delta L' = \frac{q}{V\alpha} \delta M$$

In direct flow, the total pitching moment is given by the integration of the elementary pitching moment over the plan form, the element of integration being $dx dy$; similarly, in reverse flow, the total lift is given by the integration of the elementary lift over the plan form, the element of integration being $dx dy$. This integration results in

$$L' = \frac{qM}{V\alpha}$$

When this relationship is converted to the nondimensional form

$$C_{Lq}' = \frac{\partial C_{L'}}{\partial \frac{qc}{2V}}$$

there results

$$C_{Lq}' = 2C_{m\alpha} \quad (19)$$

It should be noted in connection with the relationship expressed in equation (19), that the reference axis for C_m in direct flow and the reference axis for q in reverse flow must be fixed with respect to the wing configuration. For example, if C_m for the wing in direct flow is obtained with respect to an axis passing through the leading edge of the tip chord, then when the flow is reversed, the leading and trailing edges become interchanged. The corresponding axis for the wing in reverse flow for pitching motion q then passes through the trailing edge of the tip.

Damping in pitch C_{mq} .- If the wing in direct flow (fig. 4(a)) is pitching about the y -axis, then

$$w_0(\xi, \eta) = -q\xi \quad (20)$$

likewise, in reverse flow (fig. 4(b)),

$$w_0'(x, y) = qx \quad (21)$$

In figure 4(a) for the wing in direct flow, the elementary pitching moment δM induced by the sources in region A on the element $dx dy$ can be obtained from equation (13) by inserting the value for w_0 given by equation (20) and multiplying by the moment arm x . Thus,

$$\begin{aligned} \delta M(dx dy) &= -x\delta L(dx dy) \\ &= -2\rho V q x dx dy \left[\frac{1}{\pi} \iint_A \frac{-\xi(x - \xi)d\xi d\eta}{R^3} \right. \\ &\quad \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{-\xi d\xi}{R} + \frac{x}{B} \right] \quad (22) \end{aligned}$$

In figure 4(b) for the wing in reverse flow, the elementary pitching moment $\delta M'$ induced on region A by the sources in the element $dx dy$ is obtained from equation (14) by substituting for w_0' the relation given by equation (21) and by inserting the local moment arm ξ under integral signs. The moment arm for the last term is $\xi = x$. Thus,

$$\delta M' = 2\rho V q x dx dy \left[\frac{1}{\pi} \iint_A \frac{\xi(x - \xi) d\xi d\eta}{R^3} + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{\xi d\xi}{R} - \frac{x}{B} \right] \quad (23)$$

A comparison of equations (22) and (23) shows that the elementary pitching moments δM and $\delta M'$ are equal. For both the wing and its reverse, the total pitching moment is given by the integration of the elementary pitching moments δM and $\delta M'$ over the plan form, the element of integration being $dx dy$. Thus, the derivative C_{mq} for the wing and its reverse are the same:

$$C_{mq} = \frac{\iint_S \delta M dx dy}{\alpha \frac{\rho}{2} V^2 S \bar{c}} = C_{mq}'$$

The theorem of reversibility is thus proved for the damping in pitch. As indicated previously for the reversibility relationship for $C_{m\alpha}$ and C_{Lq} , it should be noted that the reference axes for the moments and pitching angular velocities for the wing and its reverse must be fixed with respect to the wing configuration.

Derivatives C_{L_u} and C_{m_u} . - The derivative C_{L_u} is a nondimensional form of the rate of change of lift with forward speed

$$\begin{aligned} C_{L_u} &= \frac{1}{\frac{\rho}{2} V S} \left(\frac{\partial L}{\partial V} \right)_{w=\text{Constant}} \\ &= \frac{1}{\frac{\rho}{2} V S} \frac{\partial}{\partial V} \left(\frac{\rho V^2}{2} S C_{L\alpha} \frac{w}{V} \right)_{w=\text{Constant}} \\ &= \frac{w}{V} \frac{\partial}{\partial V} (V C_{L\alpha}) \\ &= \alpha \frac{\partial}{\partial V} (V C_{L\alpha}) \end{aligned} \quad (24)$$

A preceding section that treated the derivative C_{L_α} has shown that $C_{L_\alpha}' = C_{L_\alpha}$. The remaining parameters in equation (24) are also identical for the wing and its reverse; therefore,

$$C_{L_u}' = C_{L_u}$$

The derivative C_{m_u} is a nondimensional form of the rate of change of pitching moment with forward speed:

$$\begin{aligned} C_{m_u} &= \frac{1}{\frac{\rho}{2} V S \bar{c}} \left(\frac{\partial M}{\partial V} \right)_{w=\text{Constant}} \\ &= \frac{1}{\frac{\rho}{2} V S \bar{c}} \frac{\partial}{\partial V} \left(\frac{\rho}{2} V^2 S \bar{c} C_{m_\alpha} \frac{w}{V} \right)_{w=\text{Constant}} \\ &= \alpha \frac{\partial}{\partial V} (V C_{m_\alpha}) \end{aligned} \quad (25)$$

A preceding section that treated the derivatives C_{m_α} and C_{L_q} has shown that

$$C_{m_\alpha}' = \frac{C_{L_q}}{2}$$

where the axis of rotation is fixed with respect to the wing configuration. All the remaining parameters in equation (25) are identical for the wing and its reverse. The derivative C_{m_u}' for the reversed wing can therefore be obtained from the following relationship:

$$C_{m_u}' = \frac{\alpha}{2} \frac{\partial}{\partial V} (V C_{L_q}) \quad (25a)$$

where C_{L_q} is a function of V through its dependence on Mach number. Here again C_{m_u}' and C_{L_q} refer to an axis of rotation which is fixed with respect to the wing configuration.

Derivatives C_{D_u} , C_{D_α} , and C_{D_q} .-- The reversibility relationships for the derivatives C_{D_u} , C_{D_α} , and C_{D_q} require special consideration

for a general wing of the type illustrated in figure 1 or figure 4. If the wing has subsonic side edges, the portions that are raked out will develop a suction force giving thrust along the side edge. When the flow is reversed, these portions of the side edge are raked in and become, in effect, a subsonic trailing edge, and, therefore, they cannot produce any suction force if the Kutta-Joukowski condition is applied. Consequently, the theorem of reversibility cannot be demonstrated for these cases.

However, the resultant force on a thin flat wing with a supersonic leading edge and streamwise tip acts normal to the surface as there is no suction at the leading edge. Thus, the forces in the x-direction arise solely from skin friction. On the basis that the skin friction is unchanged when the flight direction is reversed

$$C_{Du}' = C_{Du}$$

If the skin friction is assumed to be independent of α and q , the derivatives $C_{D\alpha}$ and C_{Dq} , with respect to body axes, will be zero for both the wing and its reverse.

Damping in roll C_{lp} . - If the wing in direct flow, (fig. 4(a)) is rolling about the x-axis, then

$$w_0(\xi, \eta) = -p\eta \quad (26)$$

and, in reverse flow, (fig. 4(b))

$$w_0'(x, y) = py \quad (27)$$

In figure 4(a) for direct flow, the elementary rolling moment δL_r induced by the sources in region A on the element $dx dy$ can be obtained from equation (13) by inserting the value for w_0 given by equation (26) and multiplying by the moment arm y . Thus,

$$\begin{aligned} \delta L_r(dx dy) &= -y\delta L(dx dy) \\ &= -2\rho V p y \, dx \, dy \left[\frac{1}{\pi} \left[\iint_A \frac{-\eta(x - \xi)d\xi d\eta}{R^3} \right. \right. \\ &\quad \left. \left. + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{-\eta d\xi}{R} + \frac{y}{B} \right] \right] \quad (28) \end{aligned}$$

In figure 4(b) for reverse flow, the elementary rolling moment $\delta L_r'$ induced on region A by the sources in the element $dx dy$ is obtained from equation (14) by substituting for w_0' the relation given by equation (27), and inserting the local moment arm η under the integral signs. The moment arm for the last term is $\eta = y$. Thus,

$$\delta L_r' = 2\rho V p y \, dx \, dy \left[\frac{1}{\pi} \left| \iint_A \frac{\eta(x - \xi) d\xi \, d\eta}{R^3} + \frac{1}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{\eta \, d\xi}{R} - \frac{y}{B} \right] \quad (29)$$

A comparison of equations (28) and (29) shows that the elementary rolling moments δL_r and $\delta L_r'$ are equal. For both the wing and its reverse, the total rolling moment will be given by integration of the elementary rolling moments δL_r and $\delta L_r'$ over the plan form, the element of integration being $dx dy$. Thus, the derivatives C_{l_p} for the wing and its reverse are the same:

$$C_{l_p} = \frac{\iint_S \delta L_r \, dx \, dy}{\frac{\rho}{2} V^2 S \frac{pb^2}{2V}} = C_{l_p}'$$

UNSTEADY MOTION

Derivation of Basic Equations for Unsteady Motion

The velocity potential for unsteady motion at the point (x, y) in the plane of the wing and at time t may be expressed in the following form (see reference 9 or reference 10):

$$\phi(x, y, t) = - \frac{1}{2\pi} \iint \frac{[w_0(\xi, \eta, t - \tau_1) + w_0(\xi, \eta, t - \tau_2)] d\xi \, d\eta}{R} \quad (30)$$

where τ_1 and τ_2 are the time-lag functions

$$\tau_1 = \frac{(x - \xi)M}{B^2 a} + \frac{R}{B^2 a} \quad (31)$$

$$\tau_2 = \frac{(x - \xi)M}{B^2 a} - \frac{R}{B^2 a} \quad (32)$$

and where the integration extends over the entire region in the plane of the wing that can influence the point (x,y) .

Equation (30), which gives the potential distribution with time dependency, corresponds in form to equation (1), which gives the potential distribution for the steady case. The significance of the form of equation (30) has been discussed previously following the presentation of equation (1). The region that can influence a point (x,y) in the unsteady case is contained within the Mach forecone from the point, and corresponds identically to that described previously for the steady case, as indicated in figure 2.

The present analysis considers unsteady motions in which the wing is accelerating uniformly in the vertical direction. It is shown in reference 9 that, for this type of unsteady motion in certain cases, the influence of the external field F (fig. 2) on the point (x,y) is identically canceled by the effect of the region G in the wing. This cancellation is valid under the conditions noted previously for the steady case.

The angle of attack at the beginning of the unsteady motion is assumed to be zero, inasmuch as this assumption does not result in any loss in generality because the principle of superposition is applicable. Then, for a uniform acceleration, $w_0(\xi,\eta,t) = -V\dot{\alpha}t$ for all values of ξ and η . Thus,

$$w_0(\xi,\eta,t - \tau_1) = -V\dot{\alpha}(t - \tau_1)$$

$$w_0(\xi,\eta,t - \tau_2) = -V\dot{\alpha}(t - \tau_2)$$

If these values for w_0 are substituted into equation (30), the following equation results:

$$\phi(x,y,t) = \frac{V}{\pi} \iint \frac{\dot{\alpha} \left[t - \frac{M(x - \xi)}{B^2 a} \right]}{R} d\xi d\eta \quad (33)$$

The pressure difference at the point (x,y) at time t is

$$\Delta P(x,y,t) = 2\rho V \left[\phi_x(x,y,t) + \frac{\phi_t(x,y,t)}{V} \right] \quad (34)$$

The quantity ϕ_x may be obtained by differentiating equation (33) under the integral sign with special attention to the variable limits for ξ and η . (See fig. 2.) If this differentiation is performed, the contribution of each component region for $\phi(x,y,t)$ in equation (33) results essentially in five integrals which are analogous to the result obtained for the steady case. (See also appendix A for steady case.) The quantity ϕ_t may be obtained by direct differentiation of equation (33) under the integral sign.

As a result of an analysis similar to that indicated in appendix A, the pressure difference from equations (33) and (34) can be expressed in the following form:

$$\Delta P(x,y,t) = 2\rho V^2 \alpha \left(\frac{1}{\pi} \left\{ - \iint_A \frac{\left[t - \frac{M(x-\xi)}{B^2 a} \right] (x-\xi) d\xi d\eta}{R^3} - \frac{M}{B^2 a} \iint_A \frac{d\xi d\eta}{R} + \left(\frac{\partial \eta_{2D}}{\partial x} \right) \int_a \frac{\left[t - \frac{M}{B^2 a} (x-\xi) \right] d\xi}{R} + \frac{1}{V} \iint_A \frac{d\xi d\eta}{R} \right\} + \frac{t}{B} \right) \quad (35)$$

where A and a refer to the area and line, respectively, as shown in figure 2. Letting $t \rightarrow 0$ in equation (35) in order to obtain the instantaneous value of the derivative and simplifying yields

$$\Delta P(x,y) = \frac{2\rho V \alpha}{\pi B^2} \left[M^2 \iint_A \frac{(x-\xi)^2 d\xi d\eta}{R^3} - \left(\frac{\partial \eta_{2D}}{\partial x} \right) M^2 \int_a \frac{(x-\xi) d\xi}{R} - \iint_A \frac{d\xi d\eta}{R} \right] \quad (36)$$

Elementary Lifts for Wing in Direct Flow and in Reverse Flow

Equation (36), when modified for the wake effect contributed by portions of a subsonic side edge that are raked inward, is the basic equation for deriving a relation between the lift contributed by a uniform variation of angle of attack with time for the wing and its reverse. The concept of a forezone of influence and an afterzone of influence for a point is again used in the same manner as described previously for figure 3 for the case of steady motions.

The following results are obtained by means of a development analogous to that used to obtain the elementary lifts for steady motions given previously.

Elementary lift in direct flow.- The lift on the element $dx dy$ in figure 4(a) for direct flow can be evaluated at once. The pressure difference at $dx dy$ is given by equation (36) and the lift δL is equal to $\Delta P dx dy$. Thus, since

$$\frac{\partial \eta_{2D}}{\partial x} = - \frac{1}{B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right)$$

$$\begin{aligned} \delta L(dx dy) = \frac{2\rho V \dot{\alpha}}{\pi B^2} dx dy & \left[M^2 \left| \iint_A \frac{(x - \xi)^2 d\xi d\eta}{R^3} \right. \right. \\ & \left. \left. + \frac{M^2}{B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{(x - \xi) d\xi}{R} - \iint_A \frac{d\xi d\eta}{R} \right] \end{aligned} \quad (37)$$

Elementary lift in reverse flow.- The lift on region A of figure 4(b) due to the sources in the element $dx dy$ in reverse flow is less easily obtained. As noted previously in the analysis for steady motions in connection with the reverse flow shown in figure 4(b), account must be taken of the vorticity shed in the wake of the portion of the wing tip that acts as a subsonic trailing edge. (See fig. 3(b).) The contribution of this vorticity or wake effect to $\Delta P'$ for a point on the wing in reverse flow is designated as $\Delta(\Delta P')$. This contribution $\Delta(\Delta P')$ is determined on the basis that the Kutta-Joukowski condition is satisfied along the subsonic side edge. The contribution to $\Delta P'$ of the part other than the wake effect is obtained from equation (36) with reference to figure 3(a) in a manner analogous to that used in the steady-motion case to derive equation (4) from equation (3). By following this procedure, $\Delta P'$ at the point (ξ, η) in reverse flow (fig. 3(a)) is obtained as

$$\begin{aligned} \Delta P'(\xi, \eta) = \frac{2\rho V \dot{\alpha}}{\pi B^2} & \left[M^2 \left| \iint_A \frac{(x - \xi)^2 dx dy}{R^3} \right. \right. \\ & \left. \left. + \frac{M^2}{B} \left(\frac{1 + B \tan \gamma}{1 - B \tan \gamma} \right) \int_{a'} \frac{(x - \xi) dx}{R} - \iint_A \frac{dx dy}{R} \right] + \Delta(\Delta P') \end{aligned} \quad (38)$$

An analysis for determining the wake-effect correction for uniformly accelerated motion $\Delta(\Delta P')$ is presented in appendix C. As a result of this analysis, it can be shown that, analogous to the case of steady motions for flat plates in which the side edges are subsonic and raked in, if the Kutta-Joukowski condition is applied, the vorticity in the wake in uniformly accelerated motion is of just sufficient strength to cancel the contribution to ΔP of a line integral which yields an infinite pressure along the subsonic edge. This line integral which is canceled by the wake effect is evaluated on the wing along the line a' . As a result of the analysis in appendix C, the wake-effect contribution to the pressure difference in equation (38) is found to be

$$\Delta(\Delta P') = - \frac{4\rho V \dot{a}}{\pi B^2} M^2 \frac{\tan \gamma}{1 - B \tan \gamma} \int_{a'} \frac{(x - \xi) dx}{R} \quad (39)$$

If the expression for $\Delta(\Delta P')$ in equation (39) is substituted into equation (38) and the result is simplified, the expression for $\Delta P'$ at point (ξ, η) in figure 3(a) in reverse flow becomes

$$\begin{aligned} \Delta P'(\xi, \eta) = & \frac{2\rho V \dot{a}}{\pi B^2} \left[M^2 \iint_A \frac{(x - \xi)^2 dx dy}{R^3} \right. \\ & \left. + \frac{M^2}{B} \int_{a'} \frac{(x - \xi) dx}{R} - \iint_A \frac{dx dy}{R} \right] \quad (40) \end{aligned}$$

The lift on region A due to the sources in the element $dx dy$ in reverse flow (fig. 4(b)) is obtained from equation (40) in a manner analogous to that used in the analysis for steady motions in deriving equation (12) from equation (6). Thus, the differential components of equation (40) are interpreted as elementary lifts induced by particular source elements, and these lifts are then resummed in a different fashion to give the lift induced on the region A by the sources in $dx dy$. The contribution of the two surface integrals in equation (40) can be obtained in a manner similar to that indicated in the preceding analysis for steady motions, and the contribution of the line integral can be obtained in a manner analogous to that described in appendix B. This procedure leads to the result that the total lift

induced on the region A by the sources in the element $dx dy$ in reverse flow is given by the following expression:

$$\delta L' = \frac{2\rho V \dot{\alpha}}{\pi B^2} dx dy \left[M^2 \iint_A \frac{(x - \xi)^2 d\xi d\eta}{R^3} + \frac{M^2}{B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) \int_a \frac{(x - \xi) d\xi}{R} - \iint_A \frac{d\xi d\eta}{R} \right] \quad (41)$$

Reversibility Relation for Lift in Unsteady Motion

A comparison of equations (37) and (41) indicates that they are identical; in both direct flow and reverse flow, the total wing lift will be given by the integration of the elementary lift over the plan form, the element of integration being $dx dy$. Thus, the derivative $C_{L\dot{\alpha}}$ for the wing in direct flow and in reverse flow is

$$C_{L\dot{\alpha}} = \frac{\iint_S \delta L dx dy}{\frac{\rho}{2} V^2 S \frac{\dot{\alpha} \bar{c}}{2V}} = C_{L\dot{\alpha}}$$

The theorem of reversibility is thus proved for the lift contributed by a uniform variation of angle of attack with time for wings of the general type illustrated in figure 1 or figure 4.

R E S U L T S

The reversibility relations that have been derived in this analysis were obtained for a subsonic wing-tip configuration of the type shown in figure 1 or figure 4. All the results obtained for this illustrative plan form will likewise apply to wings with streamwise tips or those with subsonic tips of arbitrary rake as long as the edges are nonre-entrant. Furthermore, as indicated in the preceding analysis, equation (2) for steady motion and equation (36) for unsteady motion may be applied also to any wing having supersonic tips of arbitrary rake since the component regions C, D, and E simply degenerate into regions of the type C and E. Therefore, the relations obtained between a wing and its reverse for the stability derivatives for the case of the streamwise tip will apply also to wings that have supersonic tips of arbitrary rake.

A complete summary of the relations between the characteristics of a wing and its reverse in supersonic flow is presented in table I. The table includes all the results obtained in the present paper and also those obtained by previous investigators. It should be noted that all the stability derivatives refer to a system of body axes which is fixed with respect to the wing configuration.

CONCLUSIONS

An investigation has been made by means of the linearized theory of the relations between the supersonic stability derivatives of a wing and its reverse obtained by interchanging the leading and trailing edges. The analysis included wings that have a supersonic leading edge, a supersonic trailing edge, and either supersonic side edges or nonre-entrant subsonic side edges that are noninteracting, that is, neither edge lies in the zone of action of the other. The investigation was made for thin flat-plate wings and included steady vertical and longitudinal motions, steady rolling and pitching, and motions in which the wing is accelerating uniformly in the vertical direction. The results of the investigation were obtained with respect to body axes and indicated the following significant conclusions:

1. The rates of change of lift and drag with angle of attack and forward speed, the damping in roll, and the rate of change of lift with a uniform variation in angle of attack with time were all equal for the wing and its reverse.

2. With respect to axes of rotation that were fixed with respect to the wing configuration, the damping in pitch was equal for the wing and its reverse; whereas, the rate of change of pitching moment with angle of attack for the wing was equal to the rate of change of lift with pitching velocity multiplied by the stream velocity for the reverse

of the wing $\left(C_{m\alpha} = \frac{C_{Lq}'}{2} \right)$.

Langley Aeronautical Laboratory

National Advisory Committee for Aeronautics

Langley Air Force Base, Va., June 15, 1949

APPENDIX A

DERIVATION OF EXPRESSION FOR ϕ_x DISTRIBUTION

The linearized velocity potential for steady motion is from equation (1)

$$\phi(x,y) = -\frac{1}{\pi} \iint \frac{w_0(\xi,\eta) d\xi d\eta}{\sqrt{(x-\xi)^2 - B^2(y-\eta)^2}} \quad (A1)$$

where the integration is to be performed over the entire area that contains singularities that can influence the point (x,y) . The regions of influence for a point (x,y) are contained within the Mach forecone from the point and are illustrated in figure 2. In figure 2, the wing leading edge is of the supersonic type; whereas the wing tip is of the subsonic type. The regions C, D, E, F, and G contain singularities that will influence the point (x,y) . In the case of region F, these singularities are located off the wing tip. It will be noted in figure 2 that, in general, the limits of integration for ξ and η for a particular region may be functions of the variables x and y .

The quantity $\phi_x(x,y)$ may be obtained by differentiation of equation (A1) under the integral sign, with special attention to the variable limits. Thus, for an arbitrary region of integration, the following expression is obtained:

$$\begin{aligned} \phi_x(x,y) = & -\frac{1}{\pi} \left\{ - \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} \frac{w_0(\xi,\eta)(x-\xi) d\xi d\eta}{[(x-\xi)^2 - B^2(y-\eta)^2]^{3/2}} \right. \\ & - \frac{\partial \eta_1}{\partial x} \int_{\xi_1}^{\xi_2} \frac{w_0(\xi,\eta_1) d\xi}{\sqrt{(x-\xi)^2 - B^2(y-\eta_1)^2}} \\ & \left. + \frac{\partial \eta_2}{\partial x} \int_{\xi_1}^{\xi_2} \frac{w_0(\xi,\eta_2) d\xi}{\sqrt{(x-\xi)^2 - B^2(y-\eta_2)^2}} \right\} \end{aligned}$$

(continued on next page)

$$\begin{aligned}
& - \frac{\partial \xi_1}{\partial x} \int_{\eta_1(x,y,\xi_1)}^{\eta_2(x,y,\xi_1)} \frac{w_0(\xi_1, \eta) d\eta}{\sqrt{(x - \xi_1)^2 - B^2(y - \eta)^2}} \\
& + \frac{\partial \xi_2}{\partial x} \int_{\eta_1(x,y,\xi_2)}^{\eta_2(x,y,\xi_2)} \frac{w_0(\xi_2, \eta) d\eta}{\sqrt{(x - \xi_2)^2 - B^2(y - \eta)^2}} \Bigg\} \quad (A2)
\end{aligned}$$

In the applications of equation (A2), the integrals become improper when the region of integration is bounded by any part of the Mach fore-cone from the point under consideration. In such cases the procedure is to alter the limits of integration for η and ξ by means of a limiting process. Thus, the limits η_1 , η_2 , ξ_1 , or ξ_2 become $\eta_1 + \epsilon$, $\eta_2 - \epsilon$, $\xi_1 + \epsilon$, or $\xi_2 - \epsilon$. The final result is then obtained by permitting ϵ to approach zero.

Equation (A2), as expressed for each region of integration, consists of one surface integral and four line integrals evaluated around the boundary of the region of integration. Consider the regions C, D, E, F, and G in figure 2. It has been indicated in the body of the paper that, for a thin flat plate, the effect of region F will be identically canceled by the effect of region G. (See discussion following equation (1).) Therefore, equation (A2) can be considered only with respect to the regions C, D, and E.

The subsequent analysis refers to equation (A2) and to figure 2. This equation and this figure have been set up for the condition that the integrations are always performed first with respect to η and then with respect to ξ . Lines are indicated by means of subscripts that refer to the limit and to the region of integration; thus, η_{2C} indicates the line along the upper limit for the C region of integration (fig. 2).

It is apparent from equation (A2) and figure 2 that the following line integrals are identical and will cancel each other:

$$\left. \begin{aligned} \frac{\partial \xi_{2C}}{\partial x} \int_{\xi_{2C}} \dots d\eta &= - \frac{\partial \xi_{1D}}{\partial x} \int_{\xi_{1D}} \dots d\eta \\ \frac{\partial \xi_{2D}}{\partial x} \int_{\xi_{2D}} \dots d\eta &= - \frac{\partial \xi_{1E}}{\partial x} \int_{\xi_{1E}} \dots d\eta \end{aligned} \right\} \quad (A3)$$

Furthermore, the integral

$$\frac{\partial \xi_{1C}}{\partial x} \int_{\xi_{1C}} \dots d\eta = 0 \quad (A4)$$

because the upper and lower limits are identical. The integral

$$I_{2E} = \frac{\partial \xi_{2E}}{\partial x} \int_{\eta_1(x,y,\xi_{2E})}^{\eta_2(x,y,\xi_{2E})} \frac{w_0(\xi_{2E}, \eta) d\eta}{\sqrt{(x - \xi_{2E})^2 - B^2(y - \eta)^2}}$$

must be evaluated as the limiting value of the integral as

$$\xi_{2E} \rightarrow x$$

and

$$\eta \rightarrow y$$

This integration yields

$$I_{2E, \xi_{2E} \rightarrow x, \eta \rightarrow y} = \frac{w_0(x, y) \pi}{B} \quad (A5)$$

Inasmuch as $\eta_{2C} = L(\xi_{1E})$ is independent of x ,

$$\frac{\partial \eta_{2C}}{\partial x} \int_{\eta_{2C}} \dots d\xi = 0 \quad (A6)$$

The integrals (see equation A2)

$$\left. \begin{aligned} & - \frac{\partial \eta_{1E}}{\partial x} \int_{\eta_{1C}} \dots d\xi, - \frac{\partial \eta_{1D}}{\partial x} \int_{\eta_{1D}} \dots d\xi \\ & - \frac{\partial \eta_{1E}}{\partial x} \int_{\eta_{1E}} \dots d\xi, - \frac{\partial \eta_{2E}}{\partial x} \int_{\eta_{2E}} \dots d\xi \end{aligned} \right\} \quad (A7)$$

are evaluated along boundaries of the Mach forecone from the point (x,y) . Along these boundaries, the denominators of the integrands of these expressions are zero. Thus, each of the integrals in the expressions (A7) gives infinity. However, the surface integral in equation (A2) also has as limits

$$\eta_{1C}, \eta_{1D}, \eta_{1E}, \text{ and } \eta_{2E} \quad (A8)$$

If the surface integral in equation (A2) is evaluated for regions C, D, and E for conditions in which w_0 is either constant, is any function of ξ , or varies linearly with η , the substitution of the limits given by (A8) will give infinities which identically cancel the infinite line integrals given by the expressions (A7).

The foregoing analysis thus shows that the evaluation of $\phi_x(x,y)$ is obtained from equation (A2) by means of the finite part of the surface integral evaluated over the regions C, D, and E, or A by the expression given by equation (A5) and by the line integral evaluated over η_{2D} or a. Thus,

$$\begin{aligned} \phi_x(x,y) = & \frac{1}{\pi} \left[\iint_A \frac{w_0(\xi,\eta)(x-\xi)d\xi d\eta}{R^3} \right. \\ & \left. - \frac{1}{\pi} \left(\frac{\partial \eta_{2D}}{\partial x} \right) \int_a \frac{w_0(\xi,\eta)d\xi}{R} - \frac{w_0(x,y)}{B} \right] \quad (A9) \end{aligned}$$

A further discussion of equation (A9) is given in the body of the paper immediately following the presentation of equation (2).

APPENDIX B

EVALUATION OF CONTRIBUTION TO LIFT ON REGION A OF LINE

INTEGRAL ALONG a' IN EQUATION (6) IN REVERSE FLOW

The contribution to the lift on region A (fig. 4(b)) of the line integral $\frac{1}{\pi B} \int_{a'} \frac{w_o'(x,y) dx}{R}$ which appears in equation (6) requires special treatment. It is convenient to transform the integral to the u, v Mach line coordinate system shown in figure 6. In this coordinate system

$$\left. \begin{aligned} u &= \frac{M}{2B}(\xi - B\eta); \quad v = \frac{M}{2B}(\xi + B\eta) \\ \xi &= \frac{B}{M}(v + u); \quad \eta = \frac{1}{M}(v - u) \end{aligned} \right\} \quad (B1)$$

The element of area in the u, v coordinate system is given by

$$J\left(\frac{\xi, \eta}{u, v}\right) du dv = \frac{2B}{M^2} du dv$$

The u, v axes coincide with the Mach lines from the origin. The line a' is a Mach line (see fig. 3(a)) that can be specified by the relation

$$v = a' = \text{Constant} \quad (B2)$$

(More rigorously, the equation for a' should specify limitations for ξ , but this specification is not needed in what follows.) In the reverse flow indicated in figure 6, the field element is denoted by $du dv$ and the source element by $dv_1 du_1$. Along the line $v_1 = a'$, $dv_1 = 0$ and, by equations (B1), $dx = \frac{B}{M} du_1$. Thus, the line integral becomes

$$\frac{1}{\pi M} \int_{a'} \frac{w_o'(u_1, a') du_1}{R} \quad (B3)$$

where R is to be regarded as expressed in terms of u , u_1 , v , and a' .

Refer now to figures 3(a) and 4(b) for reverse flow. In this reverse flow, let the line a in figure 4(b) pass through a point corresponding to P shown in figure 3(a). Then the forezone from a point lying along the line a in figure 4(b) will be identical with the forezone A' from the corresponding point P in figure 3(a). Thus, in figure 4(b), the edge v of the afterzone from a point (x,y) lying on v , in the source element $dx dy$, corresponds to the edge a' of the same afterzone which may be drawn from a point on a' in figure 3(a). The point (x,y) in the source element $dx dy$ of figure 4(b) then may be considered as lying along a' in figure 3(a). This situation is represented in Mach coordinates in figure 6. In Mach

coordinates, x,y becomes u_1,v_1 . Increments of area $\frac{2B}{M^2} du dv$ and $\frac{2B}{M^2} du_1 dv_1$ are shown at u,v and u_1,v_1 , respectively. Equation (B2) specifies line a' and line a is specified by

$$u = a \quad (B4)$$

On the basis of the foregoing considerations, the line integral (B3) gives the velocity u_0' at point P contributed ultimately by all the sources along the line a' . Those sources in the length du_1 will contribute the part

$$\delta_2 u_0'(u,v) = \frac{1}{\pi M} \frac{w_0'(u_1, a') du_1}{R} \quad (B5)$$

Note that the contribution at an interval dv removed from point P along the line a differs from that given by equation (B5) by an infinitesimal of higher order. Now consider the contribution at an interval du removed from P along the other Mach line from P (parallel to a'); the sources along a' do not contribute, but those along a line parallel to a' and displaced from it by an interval dv do contribute. Again the contribution differs from that given by equation (B5) by an infinitesimal of higher order. Accordingly, equation (B5) may be taken to give the contribution to the velocity u_0' anywhere within the element $du dv$ due to the sources in the element $du_1 dv_1$.

The increment from the line integral to the lift on the element $du dv$ is $2\rho V \left(\frac{2B}{M^2} \right) du dv$ times the velocity u_0' given by

equation (B5). The incremental lift on all the elements $du dv$ touching the line a (fig. 6) is thus

$$\delta_2 L' = \frac{2\rho V}{\pi M} \left(\frac{2B}{M^2} \right) du_1 du \int_a \frac{w_o'(u_1, a')}{R} dv$$

In the present case, a' is simply v_1 , and, as can be obtained from figure 6

$$du = \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) dv_1$$

Thus

$$\delta_2 L' = \frac{2\rho V}{\pi M} \left(\frac{2B}{M^2} \right) \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) du_1 dv_1 \int_a \frac{w_o'(u_1, v_1) dv}{R}$$

If this equation is converted to Cartesian coordinates by means of equations (B1), and since $\frac{B}{M} dv = d\xi$ along a , there results

$$\delta_2 L' = \frac{2\rho V}{\pi B} \left(\frac{1 - B \tan \gamma}{1 + B \tan \gamma} \right) dx dy \int_a \frac{w_o'(x, y) d\xi}{R} \quad (B6)$$

Equation (B6) expresses the second part of the incremental lift contributed to the region A of figure 4(b) by the sources in $dx dy$. This second part results from the line-integral term in equation (6) and, as has been seen, is localized along the line a .

APPENDIX C

EVALUATION OF WAKE-EFFECT CONTRIBUTION TO PRESSURE BASED
ON KUTTA-JOUKOWSKI CONDITION ALONG SUBSONIC RAKED-IN
SIDE EDGES IN UNIFORMLY ACCELERATED VERTICAL MOTION

A solution for the contribution of the wake effect that satisfies the Kutta-Joukowski condition or $\Delta(\Delta P')$ in equation (38) must meet the following requirements:

1. The resultant pressure difference $\Delta P'$ in equation (38) must vanish along the subsonic raked-in portions of the side edge and at all points exterior to the wing.
2. The solution $\Delta(\Delta P')$ must give zero vertical velocity at all points on the wing surface.
3. The additional potential ψ associated with $\Delta(\Delta P')$ must satisfy the linearized partial-differential equation for unsteady flow, that is,

$$-B^2 \psi_{xx} + \psi_{yy} + \psi_{zz} - \frac{2M^2}{V} \psi_{xt} - \frac{1}{a^2} \psi_{tt} = 0 \quad (C1)$$

In the analysis to follow, it will be convenient to deal at first with a wing in direct flow as shown in figure 4(a), but the side edge will be assumed to be raked in instead of raked out. Thus, the field point will be denoted by (x, y) and the source points by (ξ, η) .

A consideration of the pressure difference obtained from equations (35) and (36) shows that ΔP will be made to vanish along the subsonic side edge and at all points exterior to the wing by the addition of the term

$$\Delta(\Delta P) = - \frac{4\rho V^2 a \tan \gamma}{\pi(1 + B \tan \gamma)} \int_a \frac{\left[t - \frac{M^2}{B^2 V} (x - \xi) \right] d\xi}{R} \quad (C2)$$

In equation (C2), the integral is to be evaluated along a line corresponding to a in figure 4(a), and γ is the local slope of the side edge at the point of intersection of line a with the side edge ($-\gamma$ in fig. 4(a)). The solution given by equation (C2) fulfills

the first requirement noted previously for the vanishing of the resultant pressure difference along the subsonic edge and off the wing. It remains, therefore, to show that requirements 2 and 3 are fulfilled by the solution (C2).

Equation (C2) may be expressed in the form

$$\Delta(\Delta P) = K \left[\frac{M^2}{B^2 V} \int_a \frac{\xi}{R} d\xi + \left(t - \frac{M^2 x}{B^2 V} \right) \int_a \frac{d\xi}{R} \right] \quad (C3)$$

where

$$K = \frac{4\rho V^2 \alpha \tan \gamma}{\pi(1 + B \tan \gamma)}$$

Consider the first integral in equation (C3). This integral is identical with the integral which gives the wake-effect contribution for a steadily pitching wing when the Kutta-Joukowski condition is applied along the subsonic raked-in portions of the side edge, as expressed by equation (5). It will be noted that the notation in equation (5) is for reverse flow where the field point is denoted by (ξ, η) and the source points, by (x, y) . Thus, for a wing pitching steadily at unit velocity, $w_0' = x$ in equation (5) corresponds to $w_0 = \xi$ in the first integral in equation (C3). In order for equation (5) to represent a valid solution for the wake effect, the potential associated with this $\Delta u_0'$ contribution must give zero vertical velocity on the wing surface and must satisfy the partial-differential equation for steady motion. Therefore, the potential associated with the first integral in equation (C3) likewise gives zero vertical velocity on the wing surface and is a solution for the partial-differential equation for steady flow. If this potential satisfies the partial-differential equation for steady flow,

then $-B^2 \psi_{xx} + \psi_{yy} + \psi_{zz} = 0$; also, since this flow is independent of time, the terms in equation (C1) that involve ψ_{xt} and ψ_{tt} are zero. Thus, the potential ψ satisfies the unsteady partial-differential equation (C1).

Consider now the second term in equation (C3). The

integral $\int_a \frac{d\xi}{R}$ is identical with the integral that gives the wake-effect contribution for a wing at a constant angle of attack when the Kutta-Joukowski condition is applied along the subsonic raked-in

portions of the side edge, as expressed by equation (5). For this case, equation (5) assumes the form

$$\Delta u = F \int_a \frac{d\xi}{R} \quad (C4)$$

where F is a constant. In order that the Δu in equation (C4) be a valid solution for the wake effect, the potential associated with it must satisfy the partial-differential equation for steady flow and must give zero vertical velocity on the wing surface.

The foregoing conclusions regarding equation (C4) will be used to show that the potential associated with the pressure contribution of the second integral in equation (C3) will satisfy the partial-differential equation for unsteady motion (C1) and will also give zero vertical velocity on the wing surface. Let this potential be denoted by ψ and let the potential associated with the pressure contribution given by equation (C4) be denoted by τ . The ΔP contribution of the second integral in equation (C3) is

$$K \left(t - \frac{M^2 x}{B^2 V} \right) \int_a \frac{d\xi}{R} = 2\rho V \left(\psi_x + \frac{\psi_t}{V} \right) \quad (C5)$$

Since $\tau_x = F \int_a \frac{d\xi}{R}$ (see equation (C4))

$$\int_a \frac{d\xi}{R} = \frac{\tau_x}{F}$$

Substituting this relationship into equation (C5) results in the equation

$$\psi_x = E \left(t - \frac{M^2 x}{B^2 V} \right) \tau_x - \frac{\psi_t}{V} \quad (C6)$$

where

$$E = \frac{K}{2\rho VF}$$

Integration of equation (C6) yields

$$\psi = E \int_{x_{LE}}^x \left(t - \frac{M^2 x'}{B^2 V} \right) \tau_{x'} dx' - \frac{1}{V} \int_{x_{LE}}^x \psi_t dx' \quad (C7)$$

(where x' is an auxiliary variable of integration which replaces x) and integration of equation (C7) by parts results in

$$\psi = E \left(t - \frac{M^2 x}{B^2 V} \right) \tau + \frac{M^2}{B^2 V} \int_{x_{LE}}^x \tau dx' - \frac{1}{V} \int_{x_{LE}}^x \psi_t dx' \quad (C8)$$

Differentiation of equation (C8) yields the following results:

$$\left. \begin{aligned} \psi_{xx} &= E \left[\left(t - \frac{M^2 x}{B^2 V} \right) \tau_{xx} - \frac{M^2}{B^2 V} \tau_x \right] - \frac{1}{V} \psi_{tx} \\ \psi_{yy} &= E \left[\left(t - \frac{M^2 x}{B^2 V} \right) \tau_{yy} + \frac{M^2}{B^2 V} \int_{x_{LE}}^x \tau_{yy} dx' \right] - \frac{1}{V} \int_{x_{LE}}^x \psi_{yyt} dx' \\ \psi_{zz} &= E \left[\left(t - \frac{M^2 x}{B^2 V} \right) \tau_{zz} + \frac{M^2}{B^2 V} \int_{x_{LE}}^x \tau_{zz} dx' \right] - \frac{1}{V} \int_{x_{LE}}^x \psi_{zzt} dx' \\ \psi_{xt} &= E \tau_x - \frac{1}{V} \psi_{tt} \end{aligned} \right\} \quad (C9)$$

In the present analysis only a linear variation of α with time is considered; consequently, the potentials associated with this linear variation of α will also vary linearly with t . (See equation (33).) Thus, in this analysis, the terms that involve the second derivative of ψ with respect to t will be discarded. The partial-differential equation for unsteady motion, equation (C1), then becomes

$$-B^2 \psi_{xx} + \psi_{yy} + \psi_{zz} - \frac{2M^2}{V} \psi_{xt} = 0 \quad (C10)$$

Substituting the relations given by equation (C9) into the left-hand side of equation (C10) and rearranging the terms results in

$$\begin{aligned}
 E \left\{ t - \frac{M^2 x}{B^2 V} \right\} & \left(-B^2 \tau_{xx} + \tau_{yy} + \tau_{zz} \right) + \frac{M^2}{B^2 V} \left[\int_{x_{LE}}^x \left(-B^2 \tau_{x'x'} + \tau_{yy} + \tau_{zz} \right) dx' \right] \\
 & + \frac{M^2}{V} \tau_x + \frac{M^2}{V} \tau_x - \frac{2M^2}{V} \tau_x \left\} + \frac{B^2}{V} \psi_{tx} \right. \\
 & \left. - \frac{1}{V} \left[\int_{x_{LE}}^x \left(-B^2 \psi_{xxt} + \psi_{yyt} + \psi_{zzt} \right) dx' \right] - \frac{B^2}{V} \psi_{tx} \right. \quad (C11)
 \end{aligned}$$

Expression (C11) may be simplified by use of the fact that, as indicated previously, τ satisfies the partial-differential equation for steady motion so that

$$-B^2 \tau_{xx} + \tau_{yy} + \tau_{zz} = 0 \quad (C12)$$

If the relation given by equation (C12) is substituted into the expression (C11), all the terms in expression (C11) will vanish with the exception of the term

$$- \frac{1}{V} \left[\int_{x_{LE}}^x \left(-B^2 \psi_{xxt} + \psi_{yyt} + \psi_{zzt} \right) dx' \right] \quad (C13)$$

It is seen from equation (C10) that

$$-B^2 \psi_{xx} + \psi_{yy} + \psi_{zz} = \frac{2M^2}{V} \psi_{xt}$$

therefore

$$-B^2 \psi_{xxt} + \psi_{yyt} + \psi_{zzt} = \frac{2M^2}{V} \psi_{xtt} = 0$$

Thus, the expression (C11) vanishes, and the potential ψ satisfies the partial-differential equation for unsteady motion, equation (C1).

The contribution of the potential ψ to the vertical velocity is obtained by differentiating equation (C8) with respect to z . Thus,

$$\psi_{zz \rightarrow 0} = E \left(t - \frac{M^2 x}{B^2 V} \right) \tau_z + \frac{M^2}{B^2 V} \int_{x_{LE}}^x \tau_z dx' - \frac{1}{V} \int_{x_{LE}}^x \psi_{tz} dx' \quad (C14)$$

Differentiating equation (C8) first with respect to t and then with respect to z yields

$$\psi_t = E\tau$$

and

$$\psi_{tz} = E\tau_z \quad (C15)$$

It has been shown previously that the vertical velocity contributed by τ on the wing surface is zero, or

$$\tau_{zz \rightarrow 0} = 0 \quad (C16)$$

Then, substituting the relations (C15) and (C16) into equation (C14) results in

$$\psi_{zz \rightarrow 0} = 0 \quad (C17)$$

The expression given by equation (C2) therefore satisfies all the requirements for a valid solution. Conversion of this expression so as to apply to the reverse flow shown in figure 4(b) yields

$$\Delta(\Delta P') = - \frac{4\rho V \alpha M^2 \tan \gamma}{\pi B^2 (1 - B \tan \gamma)} \int_a^x \frac{(x - \xi) dx}{R}$$

which is equation (39) given in the body of the paper.

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TABLE I

SUMMARY OF RESULTS FOR RELATIONS BETWEEN CHARACTERISTICS OF A WING AND
ITS REVERSE IN SUPERSONIC FLOW

[All results are with respect to body axes fixed in wing; results for subsonic tips are based on application of Kutta-Joukowski condition.]

Characteristics or stability derivative	Relation between wing and its reverse	Plan-form configurations (c)	Source
$C_{D_{wo}}$	$C_{D_{wo}}' = C_{D_{wo}}$ (a)	All plan forms	References 1 to 4
$C_{L_{\alpha}}$	$C_{L_{\alpha}}' = C_{L_{\alpha}}$	A	References 2 to 4 and present paper
		B	Reference 4 and present paper
$C_{m_{\alpha}}$	$C_{m_{\alpha}}' = \frac{C_{L_q}}{2}$ (b)	A	Reference 3
		B	Present paper
$C_{D_{\alpha}}$	$C_{D_{\alpha}}' = C_{D_{\alpha}} = 0$	A, B ₁	Present paper
C_{L_q}	$C_{L_q}' = 2C_{m_{\alpha}}$ (b)	A	Reference 3
		B	Present paper
C_{m_q}	$C_{m_q}' = C_{m_q}$ (b)	A	Reference 3
		B	Present paper
C_{D_q}	$C_{D_q}' = C_{D_q}$ (b)	A, B ₁	Present paper
C_{L_u}	$C_{L_u}' = C_{L_u}$	A, B	Present paper
C_{m_u}	$C_{m_u}' = \frac{\alpha}{2} \frac{\partial}{\partial V} C_{L_q}$ (b)	A, B	Present paper
C_{D_u}	$C_{D_u}' = C_{D_u}$	A, B ₁	Present paper
C_{l_p}	$C_{l_p}' = C_{l_p}$	A	Reference 3
		B	Present paper
$C_{l_{\dot{\alpha}}}$	$C_{l_{\dot{\alpha}}}' = C_{l_{\dot{\alpha}}}$	A, B	Present paper

^aIf leading edge is subsonic, airfoil section must be symmetrical.

^bAxes for rotations for wing and its reverse are fixed with respect to wing configuration.

^cThe notations A and B have the following significance:

- A all edges are supersonic; supersonic side edges are raked arbitrarily
- B supersonic leading edge, supersonic trailing edge, subsonic side edges are nonre-entrant and noninteracting, that is, neither edge lies in the zone of action of the other; when used with the subscript 1, plan form is restricted to streamwise tips (see fig. 1).



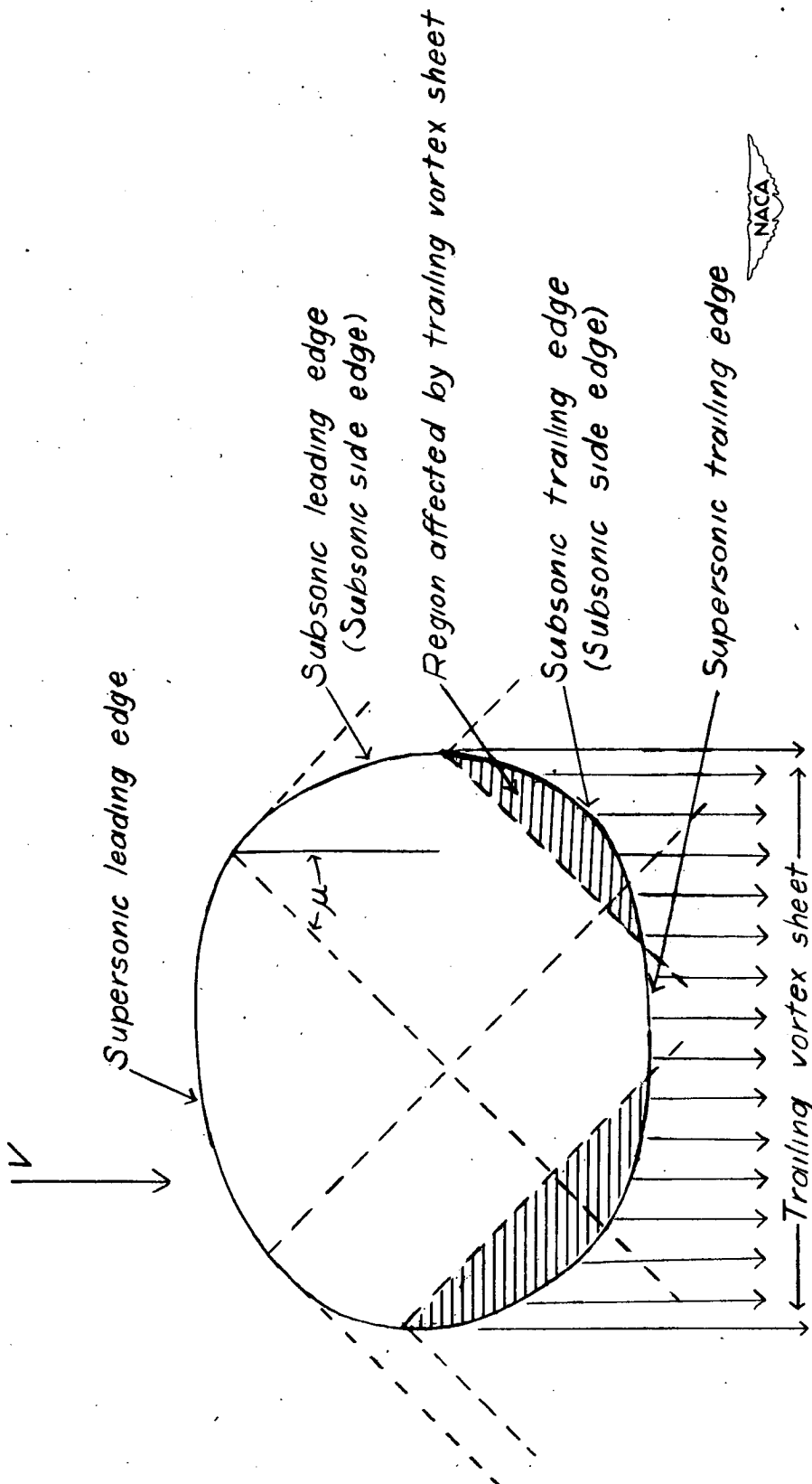
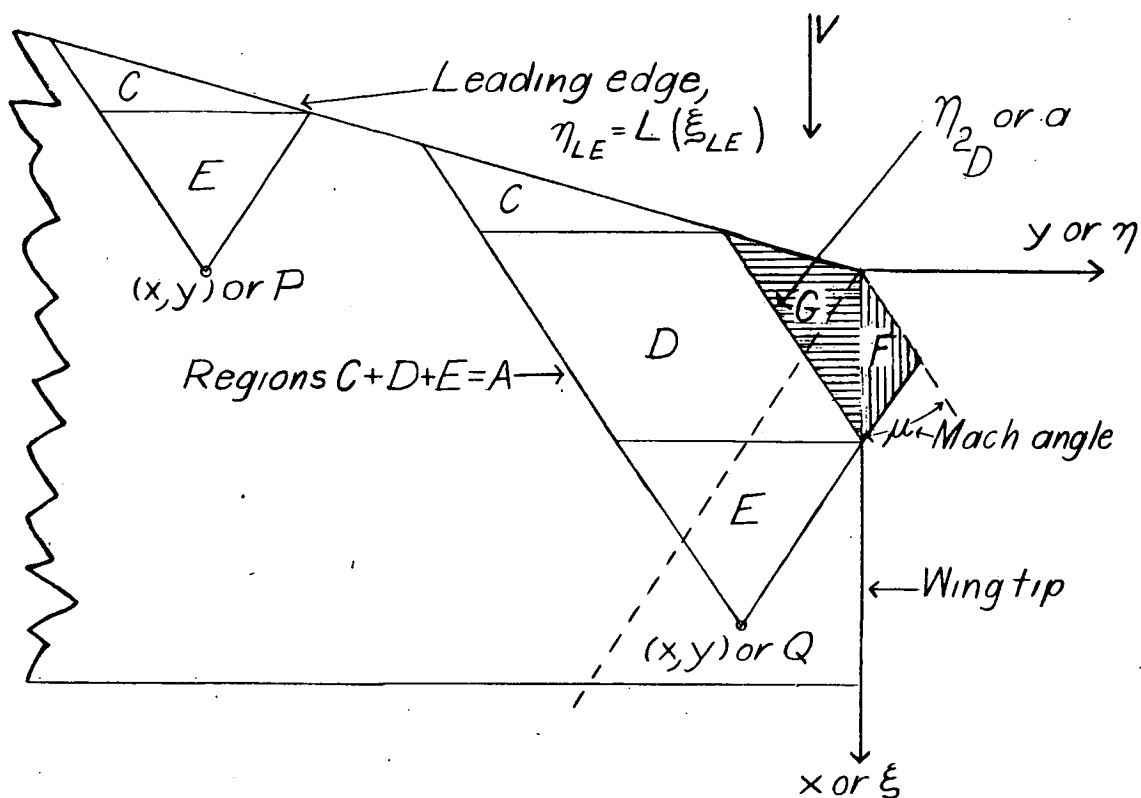


Figure 1.— General illustrative plan form for analysis. Subsonic edges do not interact.

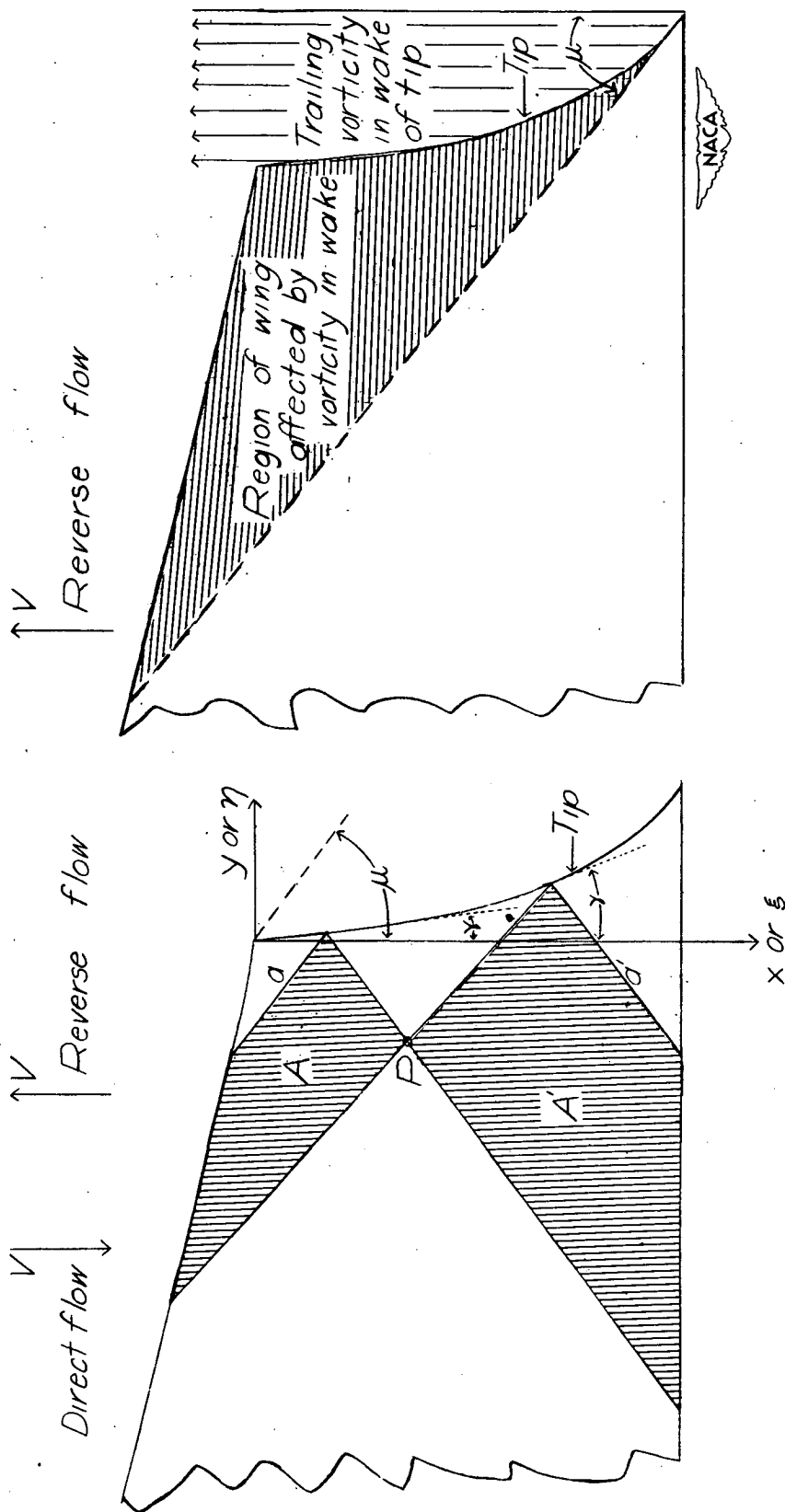


Limits of Integration When Integrations are Performed First with Respect to η and Then with Respect to ξ

Region	η	ξ
C	$\eta_1 = y - \frac{x - \xi}{B}$ $\eta_2 = L(\xi_{LE})$	$\xi_1 - BL(\xi_1) = x - By$ $\xi_2 - BL(\xi_2) = x + By$
D	$\eta_1 = \eta_{1C}$ $\eta_2 = -y - \frac{x - \xi}{B}$	$\xi_1 = \xi_{2C}$ $\xi_2 = x + By$
E	$\eta_1 = \eta_{1C}$ $\eta_2 = y + \frac{x - \xi}{B}$	$\xi_1 = \xi_{2D}$ $\xi_2 = x$



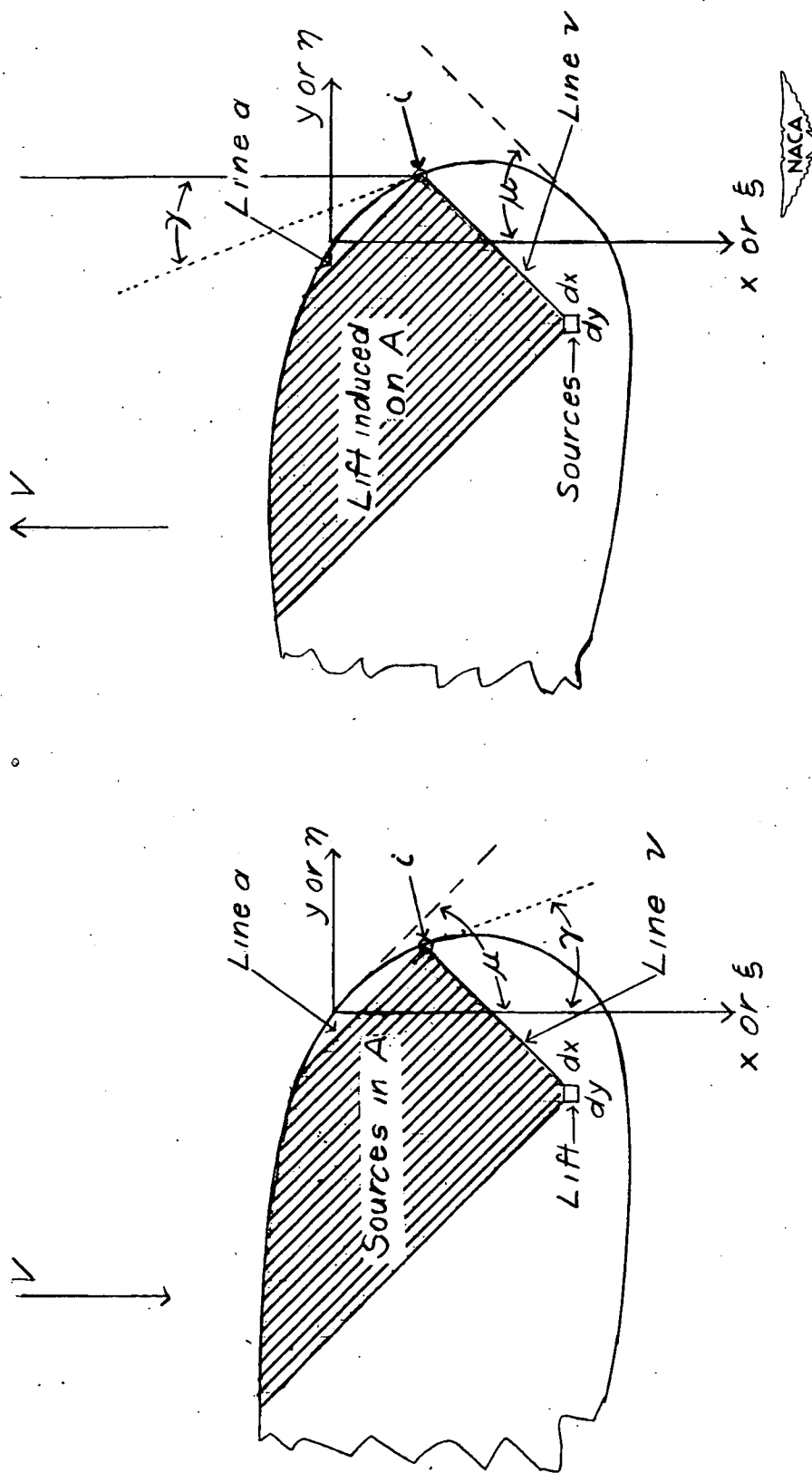
Figure 2.—Regions of influence for a point on a wing. Supersonic leading and trailing edges with streamwise wing tip.



(a) Direct and reverse flow.

(b) Region of wing affected by vorticity shed in wake of subsonic raked-in wing tip.

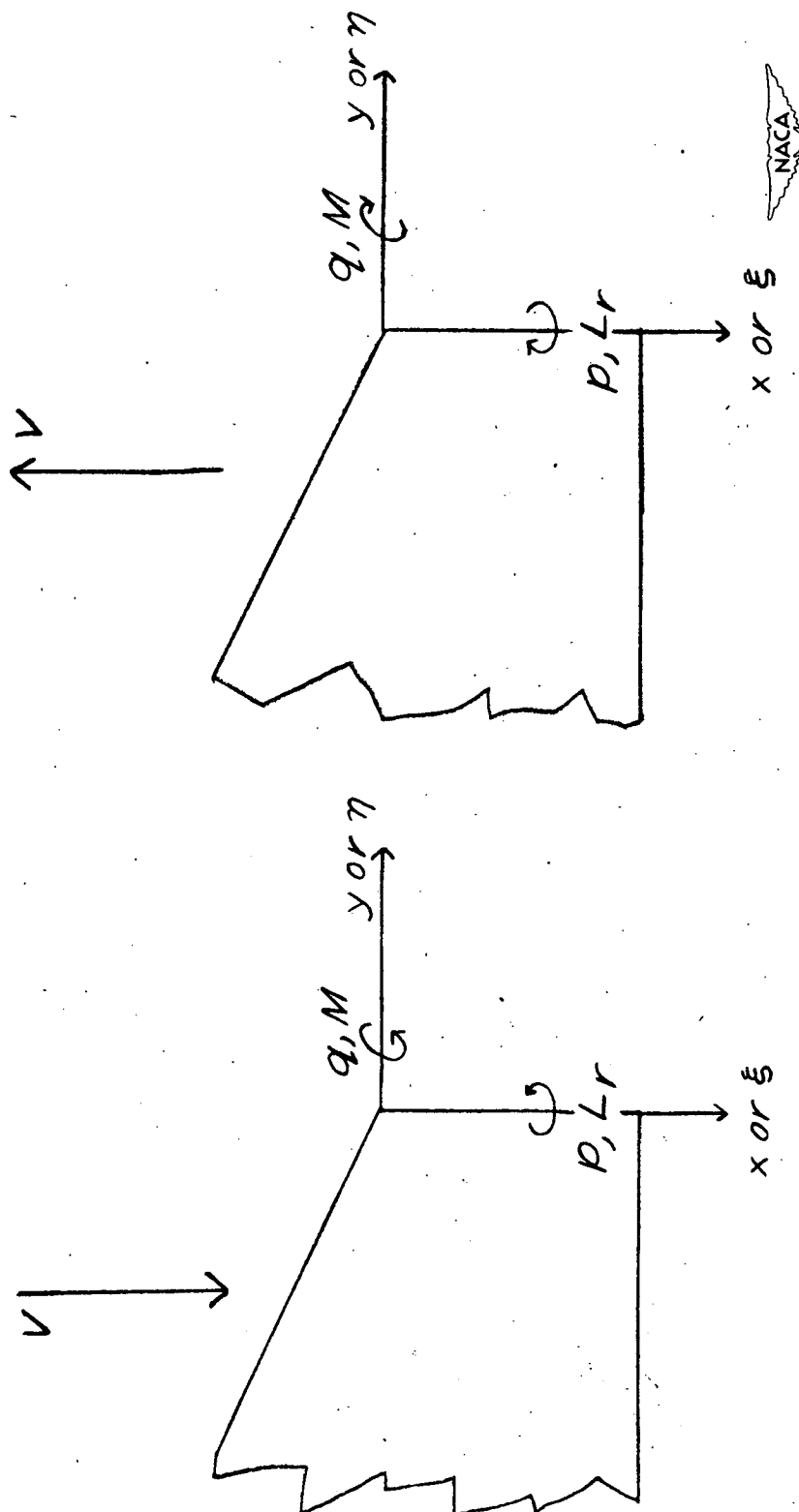
Figure 3.—Zones of influence for points in tip region. Subsonic side edge.



(b) Reverse flow.

(a) Direct flow.

Figure 4.—Elementary lift induced by sources in direct flow and in reverse flow.



(b) Reverse flow.

(a) Direct flow.

Figure 5.— Positive directions for moments and angular velocities in direct flow and in reverse flow.

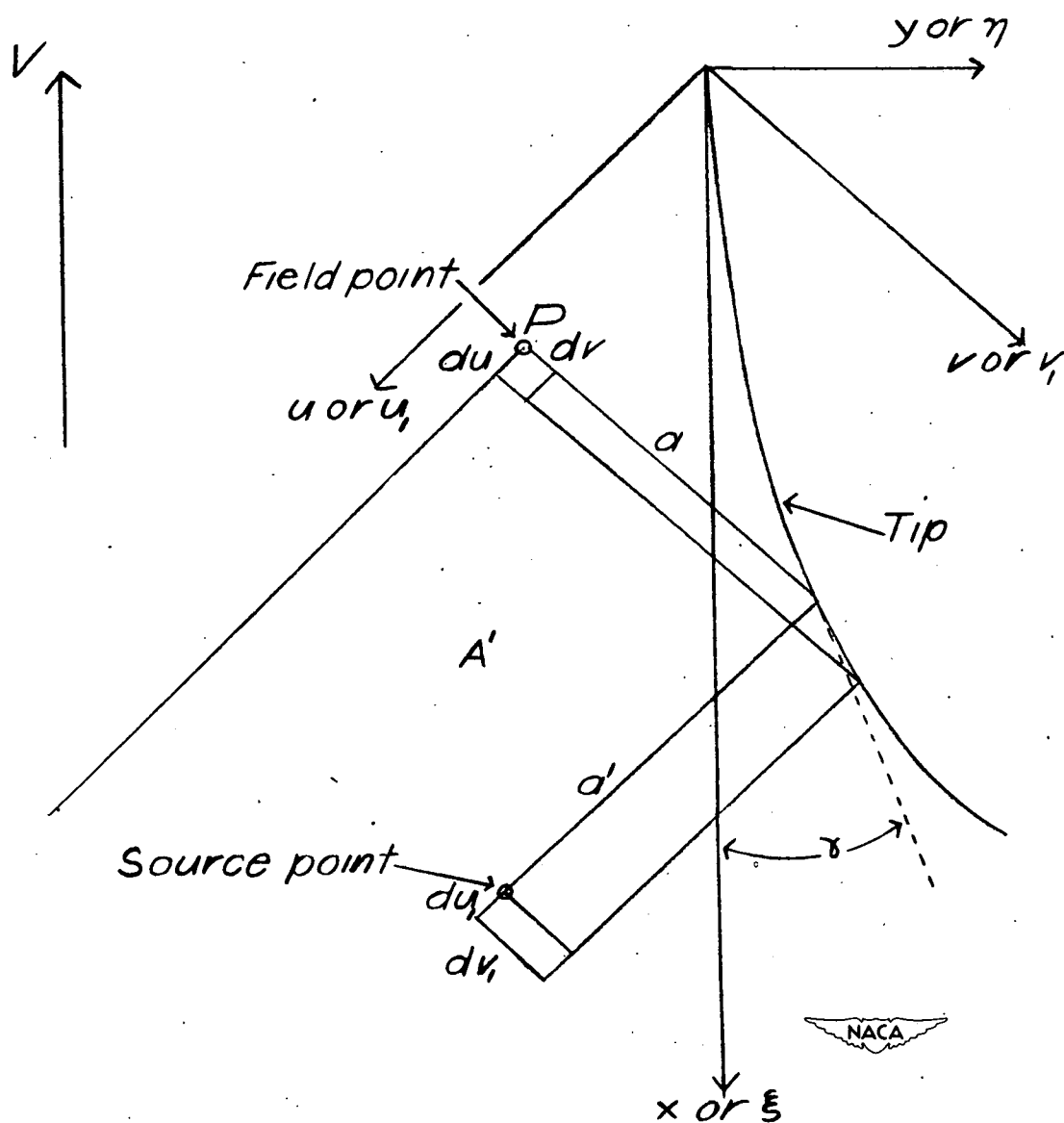


Figure 6.—Mach line coordinate system.